Implicit Methods: how to not blow up

David Baraff
Andrew Witkin
Robotics Institute

Carnegie Mellon

“Give me Stability
or
Give me Death”

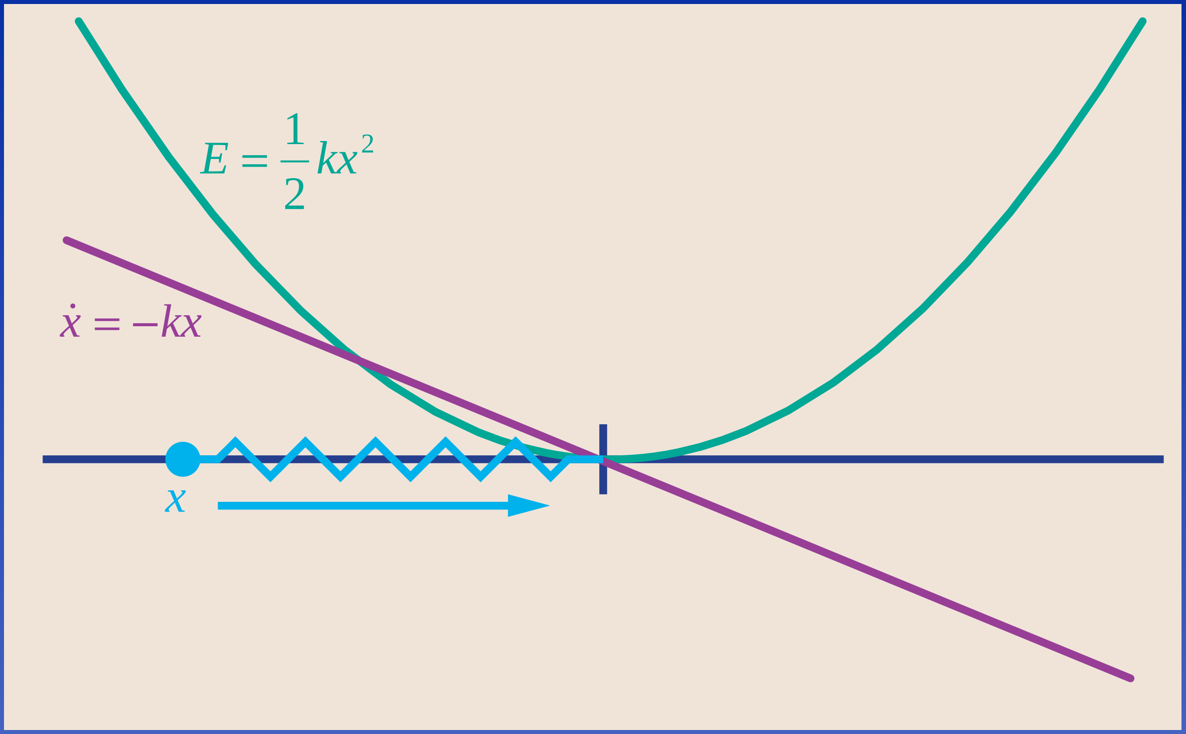
— Baraff’s other motto

stability is all stability is all stability is all

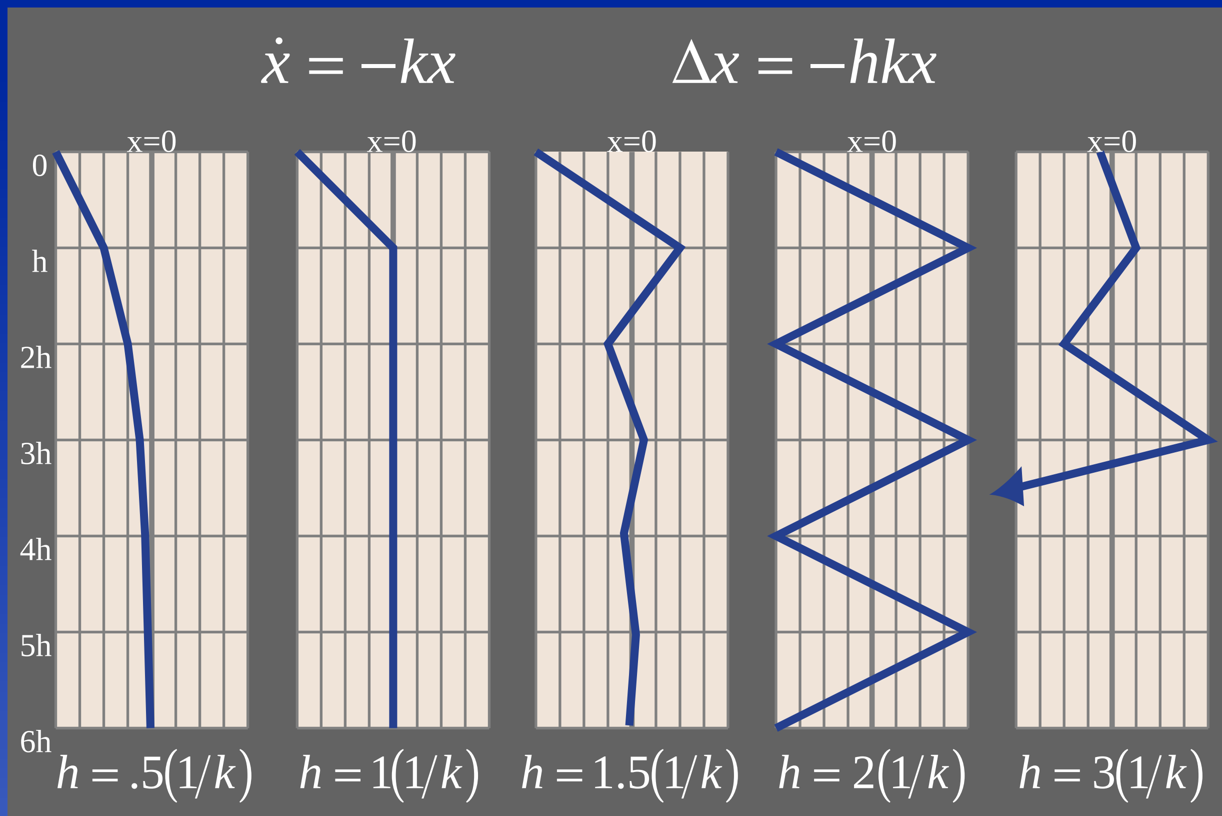
- If your step size is too big, your simulation blows up. It isn't pretty.
- Sometimes you have to make the step size so small that you never get anyplace.
- Nasty cases: cloth, constrained systems.
- Solutions:
  - –Now: use explosion-resistant methods.
  - –Later: reformulate the problem.

A very simple equation

A 1 - D particle governed by $\dot{x} = -kx$ where k is a *stiffness* constant.



Euler's method has a speed limit

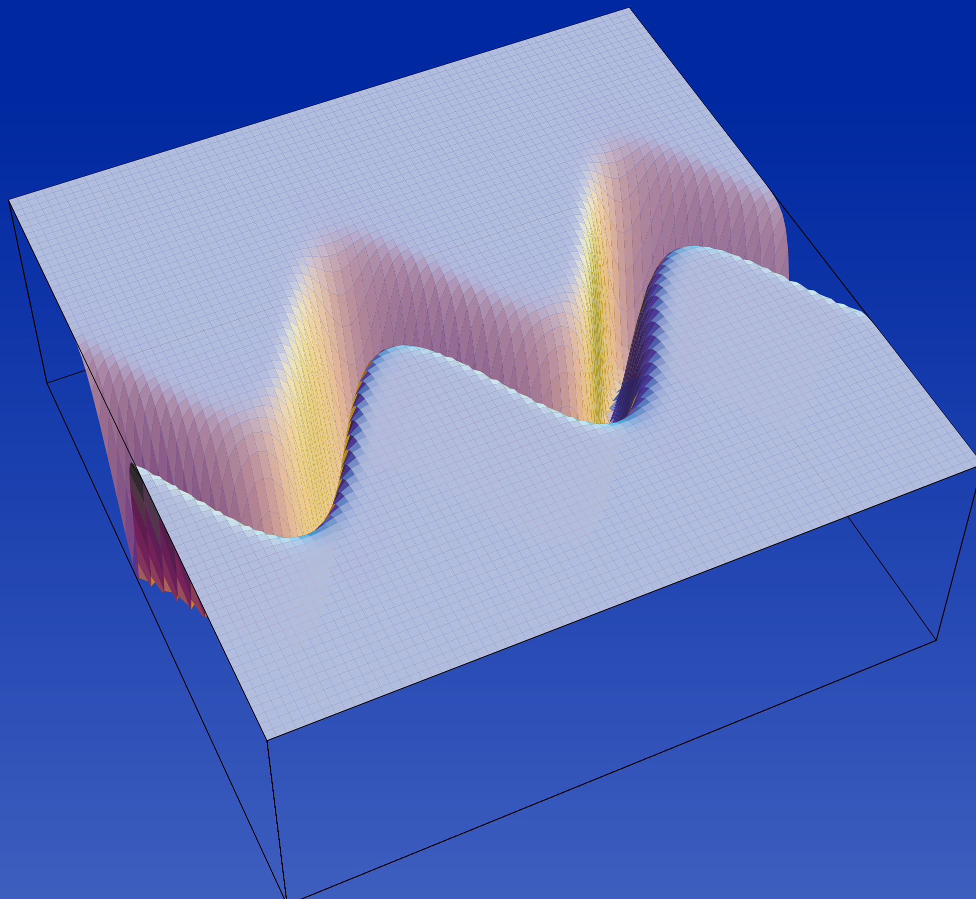


$h > 1/k$: oscillate. $h > 2/k$: explode!

Stiff Equations

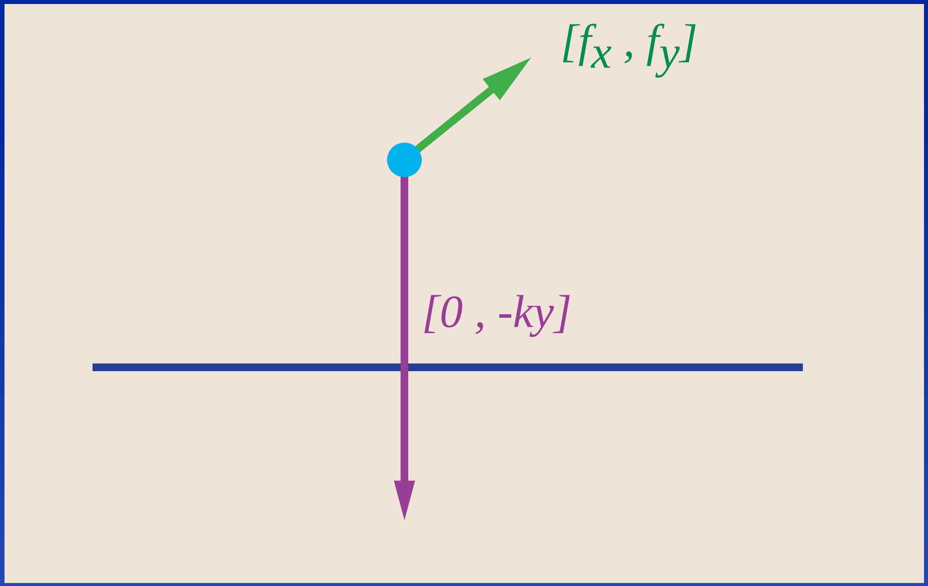
- In more complex systems, step size is limited by the largest *k*. One stiff spring can screw it up for everyone else.
- Systems that have some big *k*'s mixed in are called stiff systems.

A stiff energy landscape



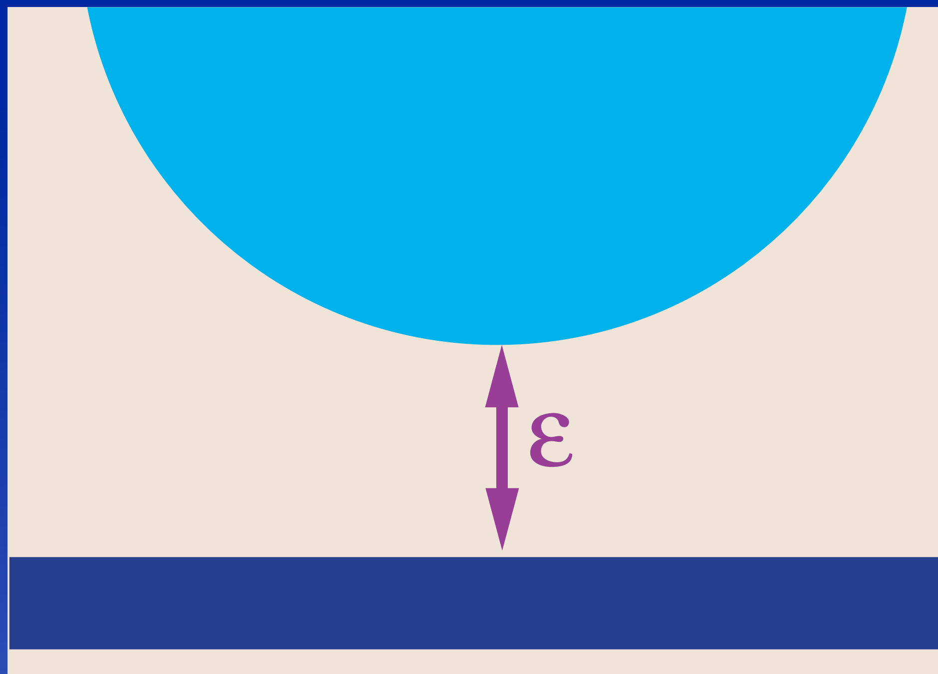
Example: particle-on-line

- A particle P in the plane.
- Interactive "dragging" force $[f_x, f_y]$.
- A penalty force $[0,-ky]$ tries to keep P on the x-axis.



- Suppose you want P to stay within a miniscule ε of the x-axis when you try to pull it off with a huge force $f_{\max}$.
- How big does k have to be? How *small* must h be?

Really big *k*. Really small *h*.



$$k = \frac{f_{max}}{\varepsilon}$$

$$h < \frac{\varepsilon}{f_{max}}$$

$$\Delta P_{max} = f_{max} h < \varepsilon$$

Answer: *h* has to be so small that *P* will never move more than ε per step.

Result: Your simulation grinds to a halt.

Implicit Methods

- *Explicit Euler*: $x(t+h) = x(t) + h\, f(x(t))$
  - –This is the version we already know about.
- *Implicit Euler*: $x(t+h) = x(t) + h\, f(x(t+h))$
  - –Evaluate the derivative at the *end* of the step instead of the beginning.
  - –Solve for $x(t+h)$.
  - –More work per step, but *much* bigger steps.
  - –A magic bullet for many stiff systems.

Implicit Euler for $\dot{x} = -kx$

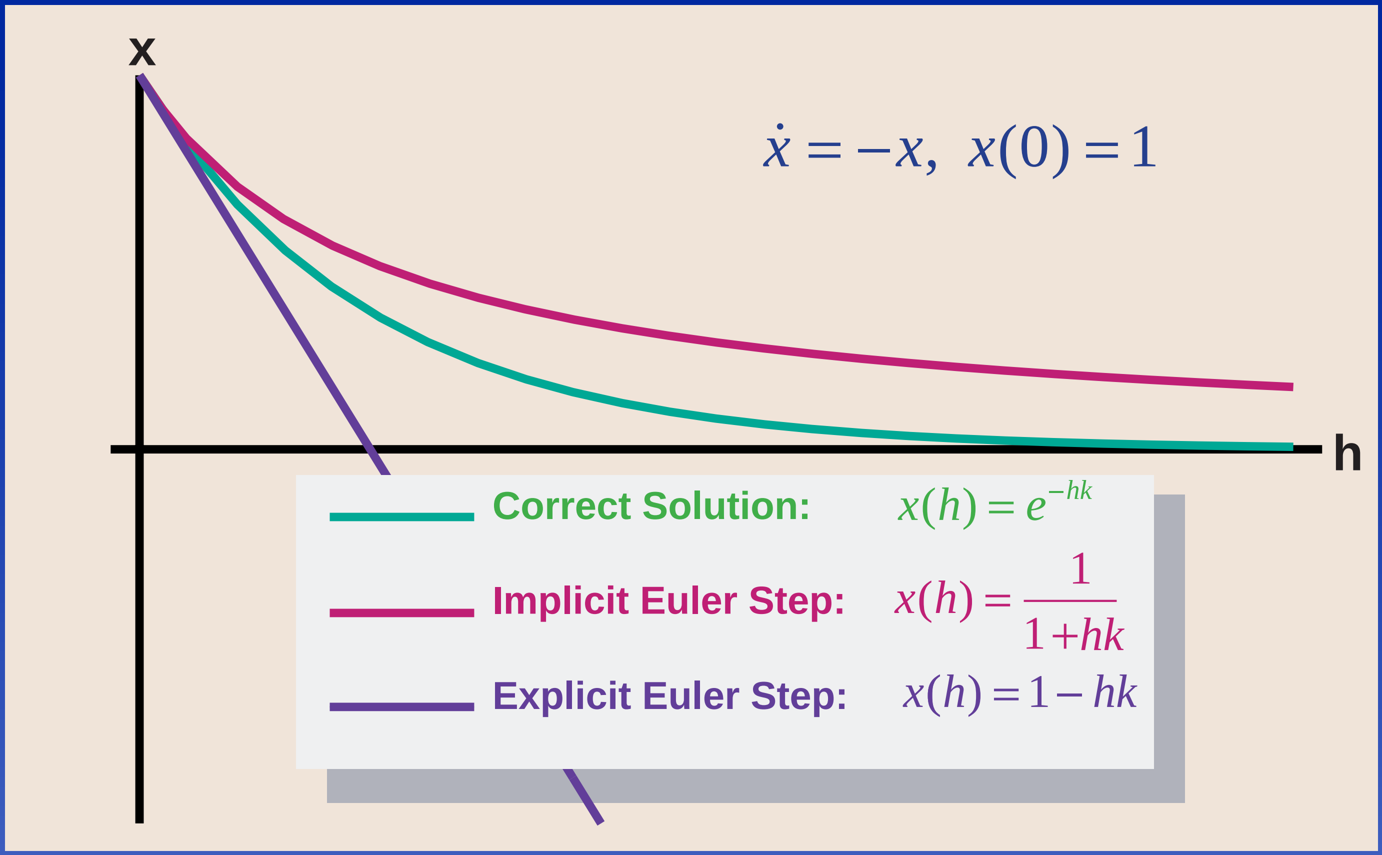
$$x(t+h) = x(t) + h\, f(x(t+h))$$

$$= x(t) - h\,k\,x(t+h)$$

$$= \frac{x(t)}{1+hk}$$

- Nonlinear: Approximate as linear, using $\partial f / \partial x$.
- Multidimensional: (sparse) matrix equation.

One Step: Implicit vs. Explicit



Why does it work?

- The real solution to $f = -kx$ is an inverse exponential.
- Implicit Euler is a decent approximation, approaching zero as h becomes large, and never overshooting. Hence, rock stable.
- Most problems aren't linear, but the approximation using $\partial f / \partial x$ —one derivative more than an explicit method—is good enough to let us take *vastly* bigger time steps than explicit methods allow.