Fourrier transform of an impulsion train

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This document was inspired by `http://math.ut.ee/~toomas_l/harmonic_analysis/` and gives the demonstration of the following theorem :

Theorem 1 (Impulsion train). *The Fourier transform of a spatial domain impulsion train of period T is a frequency domain impulsion train of frequency $\Omega = 2\pi/T$.*

$$\boxed{\sum_{p\in\mathbb{Z}} \delta(x - pT) \stackrel{FT}{\longleftrightarrow} \Omega \sum_{k\in\mathbb{Z}} \delta(x - k\Omega)} \tag{1}$$

Reminders

Fourier Coefficients

Let f be a T-periodic function, we have :

$$f(x) = \sum_{k\in Z} c_k e^{ik\Omega x} \text{ with } \begin{cases} \Omega = \dfrac{2\pi}{T} \\ c_k = \dfrac{1}{T} \displaystyle\int_0^T f(t) e^{-ik\Omega t} dt \end{cases}$$

The c_k are called *the Fourier coefficients* of f. This coefficient can be rewritten as an integral over any interval of length T. In particular, we will use :

$$\boxed{c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-ik\Omega t} dt} \tag{2}$$

Proof. Let $g(t) = f(t)e^{-ik\Omega t}$. It is a T-periodic function since we have :

$$\begin{aligned} g(t+T) &= f(t+T)e^{-ik\Omega(t+T)} = f(t)e^{-ik\Omega t}e^{-ik\Omega T} \\ &= f(t)e^{-ik\Omega t} \underbrace{e^{-ik2\pi}}_{=1} = g(t) \end{aligned}$$

Equation (2) is said due to the fact that the integral of a T-periodic function is constant over any interval of length T as can be seen from :

$$\int_c^{c+T} g(t)dt = \int_c^0 g(t)dt + \int_0^T g(t)dt + \underbrace{\int_T^{c+T} g(t)dt}_{t=u+T}$$

$$= \cancel{\int_c^0 g(t)dt} + \int_0^T g(t)dt + \cancel{\int_0^c g(u)du} = \int_0^T g(t)dt$$

□

Fourier Transform

The *Fourier transform* $F(\omega)$ of a real-valued function $f(x)$ is defined by :

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \tag{3}$$

The *inverse Fourier transform* is given by the relation

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega \tag{4}$$

When two functions are related by the Fourier transform, we note :

$$f(x) \xleftrightarrow{FT} F(\omega)$$

We have the *symmetry property* :

$$\boxed{\text{if } f(x) \xleftrightarrow{FT} F(\omega) \text{ then } F(x) \xleftrightarrow{FT} 2\pi f(-\omega)} \tag{5}$$

and the *linearity property* :

$$\boxed{\text{if } \begin{cases} f(x) \xleftrightarrow{FT} F(\omega) \\ f(x) \xleftrightarrow{FT} F(\omega) \end{cases} \quad \text{then } (\lambda f + g)(x) \xleftrightarrow{FT} (\lambda F + G)(\omega)} \tag{6}$$

The Dirac impulsion

The Dirac function $\delta(x)$ has the *sifting* property. If f is continuous at point a :

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)} \tag{7}$$

The Fourrier transform of a translated Dirac is a complex exponential :

$$\boxed{\delta(x-a) \overset{FT}{\longleftrightarrow} e^{-ia\omega}} \tag{8}$$

Impulsion train

Let's consider $it(x) = \sum_{p\in\mathbb{Z}} \delta(x-pT)$ a train of T-spaced impulsions and let's compute its Fourier transform. We first rewrite f using its Fourier coefficients :

$$it(x) = \sum_{k\in Z} c_k e^{ik\Omega x}$$

where $\Omega = 2\pi/T$. Using Eq. (2), we have :

$$\begin{aligned} c_k &= \frac{1}{T}\int_{-T/2}^{T/2} it(t)e^{-ik\Omega t}dt = \frac{1}{T}\int_{-T/2}^{T/2}\sum_{p\in\mathbb{Z}}\delta(t-pT)e^{-ik\Omega t}dt \\ &= \frac{1}{T}\sum_{p\in\mathbb{Z}}\int_{-T/2}^{T/2}\delta(t-pT)e^{-ik\Omega t}dt \end{aligned}$$

Since the function $t \mapsto \delta(t-pT)$ is null over the interval $[-T/2, T/2]$ for $p \neq 0$, we are left with only one term in the sumation :

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-ik\Omega t}dt = \frac{1}{T}\underbrace{\int_{-\infty}^{\infty}\delta(t)e^{-ik\Omega t}dt}_{=e^{-i\Omega 0}=1 \text{ by (7)}} = \frac{1}{T}$$

So finally we have an expression of the impulse train :

$$\boxed{it(x) = \frac{1}{T}\sum_{k\in\mathbb{Z}} e^{ik\Omega x}} \tag{9}$$

Applying the symmetry property (5) to the Fourier transform of a Dirac (8) we find :

$$\begin{aligned} e^{-ik\Omega(-x)} &\overset{FT}{\longleftrightarrow} 2\pi\delta(-(-x)-k\Omega) \\ e^{ik\Omega x} &\overset{FT}{\longleftrightarrow} 2\pi\delta(x-k\Omega) \end{aligned}$$

Applying linearity (6) to expression (9) we finally get the equality (1). □